Signal Analysis & Communication ECE355

Ch. 4.7 : CTFT & LCCDE

Lecture 21

26-10-2023

Ch 4.7: Continuous Time Fourier Transform And LCCDE

- As we have studied earlier, an important & useful class of CT-LTI systems satisfy a linear constant coefficient differential eqn. of the form:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y_{(t)} = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x_{(t)} \quad \text{— } (1)$$

* We consider stable LTI where $N \geq M$

- We have considered initial rest conditions for causal LTI, which is a good approximation of general LTI systems.

- Here, we will make use of CTFT to find: (I) the 'Freq. Response' and hence the 'impulse response' of the sys., (II) the 'output' of the sys.

(I)

- Recall: If $x_{(t)} = e^{st}$
  then $y_{(t)} = H_{(s)} e^{st}$

- Substituting in eqn (1) above & peforming some algebra

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} \left(H_{(s)} e^{st}\right) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} \left(e^{st}\right)$$

$$\left(\sum_{k=0}^{N} a_k s^k\right) H_{(s)} e^{st} = \left(\sum_{k=0}^{M} b_k s^k\right) e^{st}$$

$$H_{(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

* Rational Function (Ratio of polynomials in 's')

Inserting $s = j\omega$

$$H_{(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \quad \text{— } (2)$$

- And then take Inverse Fourier Transform (IFT) to get $h_{(t)}$

(If $N>1$, use Partial Fraction Expansion)

Example 1

$$a_1 y'_{(t)} + a_0 y_{(t)} = x_{(t)}$$

Using ②

$$H_{(j\omega)} = \frac{1}{a_1(j\omega)+a_0}$$

" Coefficients of RHS of LCCDE
= coefficients of LHS of LCCDE

$$= \frac{1}{a_1} \times \left(\frac{1}{a_0/a_1 + j\omega}\right)$$

$* \; e^{-at} u_{(t)} \overset{F}{\longleftrightarrow} \frac{1}{a+j\omega}$

Now IFT of $H_{(j\omega)}$ gives $h_{(t)}$

$$h_{(t)} = \frac{1}{a_1} e^{-\frac{a_0}{a_1}t} u_{(t)}$$

Example 2

$$y''_{(t)} + 4y'_{(t)} + 3y_{(t)} = x'_{(t)} + 2x_{(t)}$$

$$H_{(s)} = \frac{s+2}{s^2+4s+3}$$

* We take 's' here for simplicity only. Later we'll replace 's' by 'jω' ($s=j\omega$)

$$= \frac{s+2}{(s+1)(s+3)} \quad — Ⓐ$$

* Use Partial Fraction Expansion

Suppose:

$$H_{(s)} = \frac{A_1}{(s+1)} + \frac{A_2}{(s+3)} \quad — Ⓑ$$

To find A_1.

$$(s+1)H_{(s)} = A_1 + \frac{A_2}{(s+3)}(s+1) \Big|_{s=-1}$$

$$A_1 = (s+1)H_{(s)} \Big|_{s=-1}$$

$$= \frac{s+2}{s+3}\Big|_{s=-1} = \frac{1}{2}$$

Insert $H_{(s)}$ eqn Ⓐ

To find A_2

$$(s+3)\,H_{(s)} = \frac{A_1}{(s+1)}(s+3) + A_2 \Big|_{s=-3}$$

$$A_2 = (s+3)\,H_{(s)}\Big|_{s=-3}$$

$$= \frac{s+2}{s+1}\Big|_{s=-3} = \frac{1}{2}$$

Insert $H_{(s)}$ eqn (A)

Therefore,

eqn (B) $\Rightarrow$ $H_{(j\omega)} = \frac{1}{2} \times \frac{1}{1+j\omega} + \frac{1}{2} \times \frac{1}{3+j\omega}$ $\qquad s=j\omega$

IFT

$$h_{(t)} = \frac{1}{2}\left(e^{-t} + e^{-3t}\right)u_{(t)}$$

using wellknown CTFT Pair

$$e^{-at}u_{(t)} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$$

Ex. 1, lecture 18

(II) Finding $y_{(t)}$ from LCCDE .

- Find $H_{(j\omega)}$ as in (I)
- Find $X_{(j\omega)}$ by taking CTFT of given $x_{(t)}$
- $Y_{(j\omega)} = H_{(j\omega)}\,X_{(j\omega)}$

$\boxed{\text{* Convolution in time domain is Multiplication in freq. domain}}$

- Take IFT of $Y_{(j\omega)}$

$$F^{-1}\{Y_{(j\omega)}\} = y_{(t)}$$

Example 3

$$a_1 y'_{(t)} + a_0 y_{(t)} = x_{(t)}$$

(Example 1)

$x_{(t)} = e^{-t} u_{(t)}$, $y_{(t)} = ?$

$$h_{(t)} = \frac{1}{a_1} e^{-\frac{a_0 t}{a_1}} u_{(t)} \longleftrightarrow \frac{1}{a_1} \times \frac{1}{\frac{a_0}{a_1} + j\omega}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{LCCDE of Example 1}}$$

- $X_{(j\omega)} = \frac{1}{(1+j\omega)}$

- $Y_{(j\omega)} = \frac{1}{(1+j\omega)} \times \frac{1}{a_1} \times \frac{1}{\left(\frac{a_0}{a_1} + j\omega\right)}$ — (C)

- There could be two cases here: I. $\frac{a_0}{a_1} \neq 1$, II. $\frac{a_0}{a_1} = 1$.

I. If $\frac{a_0}{a_1} \neq 1$

let $Y_{(s)} = \frac{A_1}{s+1} + \frac{A_2}{s + \frac{a_0}{a_1}}$ — (D)

- Then,

$$A_1 = (s+1)\, Y_{(s)} \Big|_{s=-1} = \frac{1}{a_1} \times \frac{1}{\frac{a_0}{a_1} - 1} = \frac{1}{a_0 - a_1} \quad \text{from (C)}$$

$$A_2 = \left(s + \frac{a_0}{a_1}\right) Y_{(s)} \Big|_{s=-\frac{a_1}{a_0}} = \frac{1}{1 - \frac{a_0}{a_1}} \times \frac{1}{a_1} = \frac{1}{a_1 - a_0} \quad \text{from (C)}$$

- Therefore,

$$\text{(D)} \Rightarrow Y_{(j\omega)} = \frac{1}{a_0 - a_1} \times \frac{1}{j\omega + 1} + \frac{1}{a_1 - a_0} \times \frac{1}{j\omega + \frac{a_0}{a_1}}$$

- Taking IFT

$$y_{(t)} = \left(\frac{1}{a_0 - a_1} e^{-t} + \frac{1}{a_1 - a_0} e^{-\frac{a_0}{a_1}t} \right) u_{(t)}$$

II. If $\frac{a_0}{a_1} = 1.$

$$\text{eqn } 2 \Rightarrow \quad Y_{(j\omega)} = \frac{1}{a_1} \times \frac{1}{(1+j\omega)^2}$$

$$= -\frac{1}{ja_1} \frac{d}{d\omega} \left(\frac{1}{1+j\omega} \right) \qquad \frac{d}{dx}(x^{-1}) = -x^{-2}$$

$$= \frac{1}{a_1} t e^{-t} u_{(t)} \qquad -jtx_{(t)} \xleftrightarrow{F} \frac{d}{d\omega} X_{(j\omega)}$$

NOTE:

For $Re\{\alpha\} > 0$

$$\frac{t^{k-1}}{(k-1)!} e^{-\alpha t} u_{(t)} \xleftrightarrow{F} \frac{1}{(-j)^{k-1}(k-1)!} \frac{d^{k-1}}{d\omega^{k-1}} \left(\frac{1}{\alpha + j\omega} \right) = \frac{1}{(\alpha + j\omega)^k}$$

Example 4

$$y''_{(t)} + 4y'_{(t)} + 3y_{(t)} = x'_{(t)} + 2x_{(t)} \qquad \text{(Example 2)}$$

Suppose $x_{(t)} = e^{-t} u_{(t)}$, $y_{(t)} = ?$

$$H_{(s)} = \frac{s+2}{(s+1)(s+3)}$$

$$X_{(s)} = \frac{1}{s+1}$$

$$Y_{(s)} = X_{(s)} H_{(s)} = \frac{s+2}{(s+1)^2(s+3)} \quad — \text{(E)}$$

$$= \frac{A_{11}}{(s+1)} + \frac{A_{12}}{(s+1)^2} + \frac{A_2}{(s+3)} \quad — \text{(F)}$$

* root '-1' has multiplicity

$$A_2 = (s+3)\, Y_{(s)} \Big|_{s=-3}$$

$$= \frac{s+2}{(s+1)^2}\Big|_{s=-3} = -\frac{1}{4}$$

From (E)

$$A_{12} = (s+1)^2\, Y_{(s)} \Big|_{s=-1}$$

$$= \frac{s+2}{s+3}\Big|_{s=-1} = \frac{1}{2}$$

From (E)

- For A_{11}, we can show*:

*Appendix of Textbook

$$A_{11} = \frac{d}{ds}\left[(s+1)^2 Y_{(s)}\right]\Big|_{s=-1}$$

$$= \frac{d}{ds}\left[\frac{s+2}{s+3}\right]\Big|_{s=-1}$$

$$= \frac{(s+3)-(s+2)}{(s+3)^2}\Big|_{s=-1} = \frac{1}{4}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

- Therefore,

$$\text{(F)} \Rightarrow Y_{(j\omega)} = \frac{1}{4}\cdot\frac{1}{(j\omega+1)} + \frac{1}{2}\,\frac{1}{(j\omega+1)^2} - \frac{1}{4}\cdot\frac{1}{(j\omega+3)}$$

- IFT

$$y_{(t)} = \left(\frac{1}{4} e^{-t} + \frac{1}{2} t e^{-t} - \frac{1}{4} e^{-3t} \right) u_{(t)}$$

NOTE:

General Formula of Partial Fraction Expansion - Appendix of Textbook.

Fact: Any rational function of the form,

$$H_{(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \quad \text{where } M < N \text{ (practically)}$$

can be rewritten as the sum of fractions of the form,

$$\frac{A_{11}}{(s-e_1)} + \frac{A_{12}}{(s-e_1)^2} + \dots + \frac{A_{1\sigma_1}}{(s-e_1)^{\sigma_1}}$$

$$+ \frac{A_{21}}{(s-e_2)} + \frac{A_{22}}{(s-e_2)^2} + \dots + \frac{A_{2\sigma_2}}{(s+e_2)^{\sigma_2}}$$

$$+ \dots$$

where e_i is the ith root of $\sum_{k=0}^{N} a_k s^k$ & σ_i is the multiplicity of e_i.

Formula: for $1 \leq k \leq \sigma_i$

$$A_{ik} = \frac{1}{(\sigma_i - k)!} \frac{d^{\sigma_i - k}}{ds^{\sigma_i - k}} \left[(s-e_i)^{\sigma_i} H_{(s)} \right] \Bigg|_{s=e_i}$$

* If $M \geq N$, then simplify the rational function to:

$$f_{(s)} + \frac{\sum_{k=0}^{m'} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \leftarrow \text{remainder} \quad \text{where } m' < N.$$